ASYMPTOTIC BEHAVIOR OF SOLUTION OF A NONLINEAR RENEWAL EQUATION WITH DIFFUSION

Philippe Michel, Bhargav Kumar Kakumani

ASYMPTOTIC BEHAVIOR OF SOLUTION OF A NONLINEAR RENEWAL EQUATION WITH DIFFUSION

PHILIPPE MICHEL[†‡] AND BHARGAV KUMAR KAKUMANI[§]

Abstract. In this paper, we consider the nonlocal nonlinear renewal equation with diffusion under nonnegative initial data. Under some assumptions on the birth and death rates we prove the existence and uniqueness of the nonlinear renewal equation with diffusion and its corresponding adjoint equation also. In the next part, we prove the convergence of the solution to its steady state as time tends to infinity using the generalized relative entropy inequality and Poincaré Writinger type inequality.

Key words. Nonlinear renewal equation, steady state, convergence.

AMS subject classifications. (MSC2010) 92D25, 35A01, 35A02, 35B40.

1. Introduction. The structured mathematical models (age structured and size structured) describing the behavior of the cell population have been studied in [2, 3, 7] (and the references within). According to the biologists, the matter of which sites are active on various chromosomes determines the true age of a biological entity [5]. This true age is a multidimensional variable and can be determined by time since birth. We are mainly concerned about the population and not on the individuals, hence we assume that average aging in the population is measured from time since birth (renewal). Because of lots of sources of variation in the vector valued age of individuals, the population as a whole diffuse in population age variable.
Let $u(t,x)$ be the population density of cells at time t having age x. Assume that B, d are birth and death rates respectively. We are interested to study the dynamics of the following renewal equation with diffusion.

$$\left\{\begin{array}{l} u_t(t,x) + u_x(t,x) + d(x, S_1(t))u(t,x) = Cu_{xx}(t,x), \qquad t>0,\ x>0, \\ u(t,0) - Cu_x(t,0) = \displaystyle\int_0^\infty B(x, S_2(t))u(t,x)dx, \qquad t>0, \\ u(0,.) = u_0(.) > 0, \qquad u_0 \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+), \end{array}\right. \tag{1.1}$$

where

$$S_i(t) = \int_0^\infty \psi_i(x)u(t,x)dx, \quad \forall i = 1,2. \tag{1.2}$$

Where C is a positive constant of diffusion, u_0 is the initial age distribution, ψ_1, ψ_2 are the competition weight and $S(t) := (S_1(t), S_2(t))$ is the weighted population which depends on the competition weight ψ_i and population density u. The steady state equation of (1.1) is given by

$$\left\{\begin{array}{l} U'(x) + d(x, \bar{S}_1)U(x) = CU''(x), \qquad x>0, \\ U(0) - CU'(0) = \displaystyle\int_0^\infty B(x, \bar{S}_2)U(x)dx, \\ \displaystyle\int_0^\infty U(x)dx < \infty,\ \bar{S}_i = \int_0^\infty \psi_i(x)U(x)dx. \end{array}\right. \tag{1.3}$$

[†]Corresponding author
[‡]Math-Info, Ecole Centrale de Lyon, Ecully, 69130, France (E-mail: philippe.michel@ec-lyon.fr).
[§]Univeristy of Hyderabad, Hyderabad, 500046, India (E-mail: bharu4hcu@gmail.com).

and its corresponding adjoint equation reads as

$$
\begin{cases}
-\phi'(x) + d(x,\bar{S}_1)\phi(x) = C\phi''(x) + \phi(0)B(x,\bar{S}_2), \qquad x > 0, \\
\phi'(0) = 0, \\
\displaystyle\int_0^\infty \phi(x)U(x)dx = 1.
\end{cases} \tag{1.4}
$$

Throughout the paper, we assume that the functions d, B, ψ are nonnegative and continuous. Further we assume that there exists $L > 0$ such that for all $x,\ S_{\bar{a}},\ S_{\bar{b}}$ we have

$$
|B(x,S_{\bar{a}}) - B(x,S_{\bar{b}})| \le L|S_{\bar{a}} - S_{\bar{b}}|, \quad |d(x,S_{\bar{a}}) - d(x,S_{\bar{b}})| \le L|S_{\bar{a}} - S_{\bar{b}}|, \tag{1.5}
$$

$$
\frac{\partial}{\partial S_1} d(.,.) > 0,\ \frac{\partial}{\partial S_2} B(.,.) < 0, \tag{1.6}
$$

$$
0 \le \psi(x) \le \psi_M \tag{1.7}
$$

where ψ_M is a positive constant.

Equation (1.1) with $C = 0$ is popularly known as McKendrick–Von Foerster (MV) equation (see [6, 22]). There are several mathematicians who worked on the stability estimates and longtime behavior of the MV equation ([6, 9, 24] and the references therein) or MV - like (see [10, 12] for instance). In [21, 23] the authors have discussed the existence and uniqueness of a weak solution and have also proved the linear stability around the nontrivial steady state of the nonlinear renewal equation. In order to determine the longtime behavior of the solution, generalized relative entropy (GRE) plays a crucial role. The asymptotic behavior of the solution to the nonlinear MV equation has been studied in [16, 17, 20] using the GRE method. The well studied semi group theory techniques can also be applied to prove the existence, uniqueness results (see for instance [4, 8]).

The linear version of equation (1.1) with $C = 1$ has been studied in [1]. Touaoula *et. al.*, proved the existence and uniqueness of a weak solution. They have used Poincaré Writinger's type inequality to prove the exponential decay of the solution for large times to a steady state. In [18], Michel *et. al.*, considered the nonlinearity in the boundary term in equation (1.1) and proved the convergence of the solution towards the steady state problem. In [11], Kakumani *et. al.*, proved the existence and uniqueness of a weak solution with $S_1 = S_2$ and they have also proved the longtime behavior is some particular cases.

The aim of this paper is to extend the global convergence results of [16, 17, 20] (using the entropy method) to a more general class of nonlocal nonlinear diffusive MV equation than those seen in [1, 11, 18]. To do that, we first prove the existence and uniqueness of the solution of the equation in section 2. We also prove the existence and uniqueness of the corresponding steady state equation in the same section. Then, in section 3, we prove the asymptotic behavior of the solution using the GRE result. Finally, in section 4, we present some numerical simulations and we conclude.

2. Existence and uniqueness. In this section we prove existence and uniqueness result of solution to (1.1)–(1.2), (1.3) and (1.4). We use the same definition of weak solution and we follow the similar arguments which are used in [11] to prove the existence and uniqueness result to (1.1)–(1.2). We start with the following *a priori* estimate of u.

LEMMA 2.1. *Assume that $S(.) \in L^\infty_{\text{loc}}(\mathbb{R}^+) \times L^\infty_{\text{loc}}(\mathbb{R}^+)$, then there exists a unique weak solution $u \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+)\big) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ which solves (1.1). Moreover, we have $u \geq 0$, and*

$$\int_0^\infty |u(t,x)|dx \leq e^{||(B-d)_+||_\infty t} \int_0^\infty |u_0(x)|dx. \tag{2.1}$$

THEOREM 2.2. *Assume (1.5)–(1.7), then there is a unique weak solution $u \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+)\big) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ solving (1.1) - (1.2).*

Proof. To prove the existence and uniqueness of (1.1)–(1.2) we use banach fixed point theorem. First, we consider the banach space $X := C([0,T], \|\ \|_\infty)$ and T is chosen later. Let X_+ be the set of all nonnegative continuous functions on $[0,T]$, let $\vartheta := ||(B-d)_+||_\infty$. Let $S : [0,T] \times [0,T] \to \mathbb{R}^2$ be a continuous function. Then we have $u \in C\big([0,T]; L^1(\mathbb{R}^+)\big)$ which is a solution of

$$\begin{cases} u_t(t,x) + u_x(t,x) + d\big(x, S_1(t)\big)u(t,x) = Cu_{xx}, \quad t \in (0,T), \quad x > 0, \\ u(t,0) - Cu_x(t,0) = \displaystyle\int_0^\infty B\big(x, S_2(t)\big)u(t,x)dx, \;\; t \in (0,T), \\ u(0,x) = u_0(x), \;\; x > 0. \end{cases} \tag{2.2}$$

Now define a map $\Gamma : X_+ \times X_+ \to X_+ \times X_+$ by

$$S(t) \longmapsto \Big(\int_0^\infty \psi_1(x)u(t,x)dx, \int_0^\infty \psi_2(x)u(t,x)dx \Big).$$

Note that solutions of (1.1) – (1.2) are precisely fixed points of the map Γ. Therefore, it is enough to prove that Γ is a contraction map.
Let $u_1(t,x), u_2(t,x)$ be solutions of (2.2) corresponding to $S_a(t) := (S^a_1(t), S^a_2(t)), S_b(t) := (S^b_1(t), S^b_2(t))$ respectively. Then $\tilde{u} := u_1 - u_2$ satisfies

$$\begin{cases} \tilde{u}_t + \tilde{u}_x + d(x, S^a_1(t))\tilde{u} + [d\big(x, S^a_1(t)\big) - d\big(x, S^b_1(t)\big)]u_2 = C\tilde{u}_{xx}, \;\; t \in [0,T], \; x > 0, \\ \tilde{u}(t,0) - C\tilde{u}_x(t,0) = \displaystyle\int_0^\infty \big\{B\big(x, S^a_2(t)\big)\tilde{u}(t,x) + [B\big(x, S^a_2(t)\big) - B\big(x, S^b_2(t)\big)]u_2\big\}dx, \\ \tilde{u}(0,x) = 0. \end{cases}$$

Multiplying the above equation by $sgn(\tilde{u})$ and integrating in x and use (1.5) to get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_0^\infty |\tilde{u}(t,x)|dx\\
&\quad\le \int_0^\infty \big\{B\big(x,S_2^a(t)\big)|\tilde{u}|(t,x) + \big|B\big(x,S_2^a(t)\big)-B\big(x,S_2^b(t)\big)\big|u_2(t,x)\big\}dx\\
&\qquad\quad - \int_0^\infty d\big(x,S_1^a(t)\big)|\tilde{u}|(t,x)dx + \int_0^\infty \big|d\big(x,S_1^a(t)\big)-d\big(x,S_1^b(t)\big)\big|u_2(t,x)dx\\
&\quad\le \int_0^\infty \big\{B\big(x,S_2^a(t)\big)|\tilde{u}|(t,x) + L|S_2^a(t)-S_2^b(t)|u_2(t,x)\big\}dx\\
&\qquad\quad - \int_0^\infty d\big(x,S_1^a(t)\big)|\tilde{u}|(t,x)dx + L|S_1^a(t)-S_1^b(t)|\int_0^\infty u_2(t,x)dx\\
&\quad\le \vartheta\int_0^\infty |\tilde{u}(t,x)|dx + L|S_a(t)-S_b(t)|\int_0^\infty u_2(t,x)dx\\
&\qquad\quad \text{(where } |S_a(t)-S_b(t)| := |S_1^a(t)-S_1^b(t)| + |S_2^a(t)-S_2^b(t)|\text{)}\\
&\quad\le \vartheta\int_0^\infty |\tilde{u}(t,x)|dx + L\big(\sup_{0\le t\le T}|S_a(t)-S_b(t)|\big)|u_0|_{L^1}e^{\vartheta t}dx.
\end{aligned}
$$

By Gronwall's inequality we get

$$\int_0^\infty |\tilde{u}(t,x)|dx \le Lt\big(\sup_{0\le t\le T}|S_a(t)-S_b(t)|\big)|u_0|_{L^1}e^{\vartheta t},$$

and hence

$$\sup_{0\le t\le T}\int_0^\infty |\tilde{u}(t,x)|dx \le LT\big(\sup_{0\le t\le T}|S_a(t)-S_b(t)|\big)|u_0|_{L^1}e^{\vartheta T}.$$

Therefore from the definition of Γ and using (1.7) we obtain

$$
\begin{aligned}
\sup_{0\le t\le T}|\Gamma(S_a)-\Gamma(S_b)| \quad &\le \psi_M \sup_{0\le t\le T}\int_0^\infty |\tilde{u}(t,x)|dx\\
&\le L\psi_M T\big(\sup_{0\le t\le T}|S_a(t)-S_b(t)|\big)|u_0|_{L^1}e^{\vartheta T}.
\end{aligned}
$$

Now choose T such that Γ is a strict contraction map in a Banach space $X_+\times X_+$. Since T does not depend on the iteration process, we repeat the same process for $[T,2T],[2T,3T],....$ to obtain that there exists a unique solution u to (1.1)–(1.2) in $C\big(\mathbb{R}^+;L^1(\mathbb{R}^+)\big)$. Therefore we get that $S_i(.)$ is continuous and hence $S_i(.)\in L^\infty_{loc}(\mathbb{R}^+)$. Hence there exists a unique weak solution $u\in C\big(\mathbb{R}^+;L^1(\mathbb{R}^+)\big)\cap L^2_{loc}\big(\mathbb{R}^+;W^{1,2}(\mathbb{R}^+)\big)$ to the system (1.1)–(1.2). ☐

Now we prove the existence and uniqueness of (1.3) and (1.4). First we observe that for a given S, we have to consider the associated eigenvalue problem of (1.3) and (1.4).

Proposition 2.3. *Assume (1.6) and further assume that*

$$m \le \frac{\psi_1(x)}{\psi_2(x)} \le M, \quad 0<m\le M<\infty, \tag{2.3}$$

and

$$B(x,0)-d(x,0)>0 \quad B(x,0)-d(x,\infty)<0, \tag{2.4}$$

are satisfied then there exists a solution to (1.3)-(1.4). Moreover, if $m = M = 1$ then we have directly uniqueness of the solution.
Before we prove the Proposition 2.3, we prove some lemmas which are helpful to prove this proposition. We notice that for a given $\bar{S}$ there exists $(\lambda_{\bar{S}}, U_{\bar{S}}, \phi_{\bar{S}})$ solution to the eigenproblem (see [1] for details),

$$\begin{cases} \partial_x U_{\bar{S}} = C\Delta U_{\bar{S}} - d(x,\bar{S}_1)U_{\bar{S}} - \lambda_{\bar{S}} U_{\bar{S}}, \\ U_{\bar{S}}(0) - CU'_{\bar{S}}(0) = \displaystyle\int B(x,\bar{S}_2)U_{\bar{S}}(x)dx, \quad U_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ -\partial_x \phi_{\bar{S}} = C\Delta\phi_{\bar{S}} - d(x,\bar{S}_1)\phi_S + \phi_{\bar{S}}(0)B(x,\bar{S}_2) - \lambda_{\bar{S}}\phi_{\bar{S}}, \ \phi_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ \phi'_{\bar{S}}(0) = 0 \quad \textit{and} \quad \displaystyle\int \phi_{\bar{S}} U_{\bar{S}}(x)dx = 1. \end{cases} \tag{2.5}$$

Lemma 2.4. *Assume (1.6) then we have*

$$\begin{pmatrix} \frac{\partial}{\partial \bar{S}_1}\lambda \\ \frac{\partial}{\partial \bar{S}_2}\lambda \end{pmatrix} = \begin{pmatrix} -\int(\frac{\partial}{\partial \bar{S}_1}d)U_{\bar{S}}\phi_{\bar{S}}dx \\ \phi_{\bar{S}}(0)\int(\frac{\partial}{\partial \bar{S}_2}B)U_{\bar{S}}dx \end{pmatrix} < \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

Proof. The proof goes in similar lines that are given in [14, 15]. Therefore we skip the proof. ∎

Lemma 2.5. *Assume (1.6), $\lambda_{(0,0)} > 0$, $\lambda_{(0,\infty)} < 0$ and (2.3) then there exists a solution to (1.3).*

Proof. Using that $\lambda_{(0,0)} > 0$, $\lambda_{(0,\infty)} < 0$ and the decay (2.6), we have the existence of Γ decreasing regular function defined on $[0, \bar{S}_1^*[$ (with $\bar{S}_1^* \in [0,\infty]$) so that $\{\bar{S} = (\bar{S}_1, \bar{S}_2)\ :\ \lambda_{\bar{S}} = 0\} = \{(\bar{S}_1, \Gamma(\bar{S}_1)) :\ \bar{S}_1 \in \mathbb{R}_+\} \subset \mathbb{R}_+^2$ (1-dimension manifold). Let

$$\tilde{U}_{(\bar{S}_1,\bar{S}_2)} = \bar{S}_1 \frac{U_{(\bar{S}_1,\bar{S}_2)}}{\int U_{(\bar{S}_1,\bar{S}_2)}(x)\psi_1(x)dx},$$

and

$$\beta : \bar{S}_1 \mapsto \int \tilde{U}_{(\bar{S}_1,\bar{S}_2)}(x)\psi_2(x)dx,$$

then by assumption (2.3), we have that $m\bar{S}_1 \le \beta(\bar{S}_1) \le M\bar{S}_1$, for all $\bar{S}_1$, therefore there exists $\bar{S}_1^+$ and $\bar{S}_1^-$ such that

$$\Gamma(\bar{S}_1^+) = M\bar{S}_1^+, \quad \Gamma(\bar{S}_1^-) = m\bar{S}_1^-.$$

Moreover, for $\bar{S}_1$ small enough $(\bar{S}_1, \beta(\bar{S}_1))$ belongs to the connex set of boundary

$$\{(\bar{S}_1, M\bar{S}_1)\ :\ \bar{S}_1 \in [0, \bar{S}_1^+]\} \cup \{(\bar{S}_1, m\bar{S}_1)\ :\ \bar{S}_1 \in [0, \bar{S}_1^-]\} \cup \{(\bar{S}_1, \Gamma(\bar{S}_1)), \quad \bar{S}_1 \in [\bar{S}_1^+, \bar{S}_1^-\},$$

and for $\bar{S}_1$ large enough $(\bar{S}_1, \beta(\bar{S}_1))$ belongs to the connex set of boundary (see Figure 1)

$$\{(\bar{S}_1, M\bar{S}_1)\ :\ \bar{S}_1 > \bar{S}_1^+\} \cup \{(\bar{S}_1, m\bar{S}_1)\ :\ \bar{S}_1 > \bar{S}_1^-\} \cup \{(\bar{S}_1, \Gamma(\bar{S}_1)), \quad \bar{S}_1 \in [\bar{S}_1^+, \bar{S}_1^-\}.$$

Therefore, there exists $(\bar{S}_1, \bar{S}_2)$ such that $\int \tilde{U}_{(\bar{S}_1,\bar{S}_2)}(x)\psi_1(x)dx = \bar{S}_1$ and $\int \tilde{U}_{(\bar{S}_1,\bar{S}_2)}(x)\psi_2(x)dx = \bar{S}_2, i.e.$, a solution to (1.3). ∎

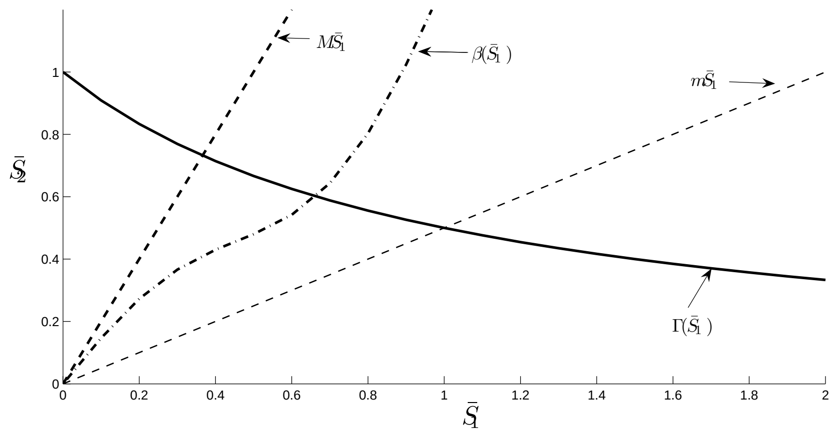

Fig. 1: $\bar{S}_1 \to \Gamma(\bar{S}_1)$ and $\bar{S}_1 \to \beta(\bar{S}_1)$. We see there exists $\bar{S}_1$ such that $\beta(\bar{S}_1) = \Gamma(\bar{S}_1)$

Remark 2.1. *It is easy to check that the Proposition 2.3 is an immediate consequence of Lemma 2.5. Notice that using (2.4), we have* $\lambda_{(0,0)} > 0$, $\lambda_{(0,\infty)} < 0$ *are satisfied.*

Uniqueness of U: Let U solution of the eigenproblem given by the Proposition 2.3 and V an another positive solution to

$$V' + d(x, \int V\psi)V = CV'', \quad V(0) - V'(0) = \int B(x, \int V\psi)V(x)dx$$

with $\int V\psi \neq \int U\psi$. Then there exists $\bar{V}, \bar{\phi}, \bar{\lambda}$ solution to the eigenproblem

$$\bar{V}' + d(x, \int V\psi)\bar{V} = C\bar{V}'' - \bar{\lambda}\bar{V}, \quad \bar{V}(0) - \bar{V}'(0) = \int B(x, \int V\psi)\bar{V}(x)dx$$

$$-\bar{\phi}' + d(x, \int V\psi)\bar{\phi} = C\bar{\phi}'' - \bar{\lambda}\bar{\phi} + B(x, \int V\psi)\phi(0), \quad \bar{\phi}'(0) = 0$$

with $\bar{\lambda} \neq 0$ (since $\frac{\partial}{\partial S}\lambda_S < 0$). Therefore by integration, we have $\bar{\lambda}\int V\bar{\phi} = 0$ and $V = 0$.

3. Asymptotic Behavior. In this section, we prove that the solution of (1.1) converges to the solution of (1.3) for large time. To prove this convergence we need additional hypothesis on ϕ, B and d, which are given below.

Definition 3.1. *Let*

$$f = u/U,$$

$$d\nu(x) = U(x)\phi(x)dx,$$

and

$$\langle f \rangle_\nu := \int f(t,x)d\nu(x) = \int u(t,x)\phi(x)dx$$

The first assumption consists in the comparison between ψ_i and ϕ :

$$k\psi_i(x) \le \phi(x) \le K\psi_i(x), \quad 0 < k \le 1 \le K < \infty \textit{ and } \int \psi_i(x)U(x)dx = 1, \; i = 1,2. \tag{3.1}$$

Therefore we chose $\bar{S}_i = 1$, $i = 1,2$. Further, the vital rates should be of the form:

$$B(x,S_1) = \begin{cases} B(x,S_2) & x \in (0,x_1], \\ B(x) & x \in (x_1,\infty). \end{cases} \tag{3.2}$$

$$d(x,S_1) = \begin{cases} d(x,S_1) & x \in (0,x_2], \\ d(x) & x \in (x_2,\infty). \end{cases} \tag{3.3}$$

for a fixed $x_1,\; x_2$ in $(0,\infty)$.

Remark 3.1. *Under the assumptions (3.2) and (3.3), we have Poincaré Writinger type inequality (see [1] for an unbounded domain) i.e.,*

$$\int_0^\infty (f - \langle f\rangle_\nu)^2 d\nu \le C_3 \int_0^\infty \Big(\frac{\partial f}{\partial x}\Big)^2 d\nu,$$

where C_3 depends on $C, \max B, \min B, \max d, \min d$.

Moreover, as in previous work (see [11]) we add the assumption (to prove the convergence of the linear case)

$$\begin{aligned} &\exists C_0,\; C_1,\; C_2 \;\; s.t. \;\; B(x,.) \ge C_0\phi(x)/\phi(0), \\ &\qquad \sup_{S_2} |\frac{\partial B}{\partial S_2}|(x) < C_1\phi(x)/\phi(0) \;\forall x \in \mathbb{R}^+ \text{ and } \sup_{x,S_1} |\frac{\partial d}{\partial S_1}| < C_2, \end{aligned} \tag{3.4}$$

and finally, we need that for a given diffusion constant C, the variation of B and d are small enough, *i.e.*,

$$r := \max\Big\{\frac{2C_1\sqrt{C_3}\tilde{K}}{k\sqrt{C_0 C}}, 2C_2\frac{\tilde{K}}{Ck\sqrt{(1-(\frac{2C_2C_3\tilde{K}}{kC})^2)}}, \frac{2C_2}{Ck}\sqrt{\tilde{K}^2 + \Big(\frac{2C_1\sqrt{C_3}\tilde{K}^2}{k\sqrt{C_0C}}\Big)^2}\Big\} < \frac{1}{2}, \tag{3.5}$$

where $\tilde{K} := \max(K, \langle f\rangle_\nu(t=0))$.

Theorem 3.2. *Under assumption (3.1)- (3.5) we have*

$$u(t,.) \to U(.), \;\; \text{as } t \to \infty \text{ in } L^2_{d\nu}(\mathbb{R}_+).$$

Proof of the main Theorem 3.2. We first state the GRE computation (see [11, 13, 17]).

Lemma 3.3. *Assume that u solves (1.1) then we have*

$$\begin{aligned} \frac{d}{dt}\langle f\rangle_\nu &= \int [d(x,1) - d(x,S_1(t))] f(t,x) d\nu(x) \\ &\qquad - \phi(0)\int f[B(x,1) - B(x,S_2(t))]U(x)dx, \end{aligned} \tag{3.6}$$

and

$$\frac{d}{dt}\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x) \tag{3.7}$$
$$= -2C\int (f')^2 d\nu(x) - \phi(0)\int [f(t,x)-f(t,0)]^2 B(x,1)U(x)dx$$
$$- 2\phi(0)[f(t,0)-\langle f\rangle_\nu]\int f[B(x,1)-B(x,S_2(t))]U(x)dx$$
$$- 2\int f[d(x,S_1(t))-d(x,1)](f-\langle f\rangle_\nu)d\nu(x).$$

Proof. By computation (multiplying equation (1.1) by ϕ and integrating with respect to x), we find that

$$\frac{d}{dt}\int f(t,x)d\nu(x)$$
$$= \int[-\partial_x u + C\Delta u - d(x,S_1(t))u]\phi(x)dx$$
$$= \int[\partial_x\phi(x) + C\Delta\phi(x) - d(x,S_1(t))\phi(x)]u(t,x)dx + \phi(0)\int B(x,S_2(t))u(t,x)dx$$
$$= \int[d(x,1)-d(x,S_1(t))]f(t,x)d\nu(x) - \phi(0)\int f[B(x,1)-B(x,S_2(t))]U(x)dx,$$

and for the computation of (3.7) refer [20] (or) see appendix (Lemma 5.2). □

Lemma 3.4. *Assume (3.1) then for all $f \in L^1_{d\nu}(\mathbb{R}_+,\mathbb{R}_+)\cap W^{1,2}_{d\nu}(\mathbb{R}_+,\mathbb{R}_+)$ we have*

$$k\int f(x)\psi_i(x)U(x)dx \le \langle f\rangle_\nu \le K\int f(x)\psi_i(x)U(x)dx, \tag{3.8}$$

$$k\int (f'^2)\psi_i(x)U(x)dx \le \int (f')^2 d\nu(x) \le K\int (f'^2)\psi_i(x)U(x)dx. \tag{3.9}$$

Proof. The proof is direct consequence under the assumption (3.1). □

Lemma 3.5. *Assume (1.6) then we have*

$$\min(k,\langle f\rangle_\nu(t=0)) \le \langle f\rangle_\nu \le \max(K,\langle f\rangle_\nu(t=0)), \quad \forall t\ge 0. \tag{3.10}$$

$$\min(k,\langle f\rangle_\nu(t=0))/K \le \int f(t,x)\psi_i(x)dx \le \max(K,\langle f\rangle_\nu(t=0))/k, \quad \forall t\ge 0. \tag{3.11}$$

$$\limsup_{t\to\infty}|\langle f\rangle_\nu - 1| \le \max_i \limsup_{t\to\infty}|\langle f\rangle_\nu - S_i|. \tag{3.12}$$

Proof. Using (3.8), if $\langle f\rangle_\nu \ge K$ then $S_i \ge 1$ (resp. if $\langle f\rangle_\nu \le k$ then $S_i \le 1$) and by equation (3.6) and (1.6) we have $\langle f\rangle_\nu$ is decreasing (respectively $\langle f\rangle_\nu$ is increasing). Now, using Lemma 3.4 we find the inequalities (3.10) and (3.11).
In order to prove the inequality (3.12), define $\epsilon_i(t) := \langle f\rangle_\nu - S_i(t)$ then

$$
\begin{aligned}
\frac{d}{dt}\langle f\rangle_\nu \quad &= \int [d(x,1)-d(x,\langle f\rangle_\nu-\epsilon_1(t))]f(t,x)d\nu(x) \\
&\qquad -\phi(0)\int f[B(x,1)-B(x,\langle f\rangle_\nu-\epsilon_2(t))]U(x)dx \\
&= ((1+\epsilon_1(t))-\langle f\rangle_\nu)\int \Big|\tfrac{d(x,1)-d(x,\langle f\rangle_\nu-\epsilon_1(t))}{((1+\epsilon_1(t))-\langle f\rangle_\nu)}\Big| f(t,x)d\nu(x) \\
&\qquad +((1+\epsilon_2(t))-\langle f\rangle_\nu)\int \Big|\tfrac{B(x,1)-B(x,\langle f\rangle_\nu-\epsilon_2(t))}{((1+\epsilon_2(t))-\langle f\rangle_\nu)}\Big| f(t,x)U(x)dx.
\end{aligned}
$$

therefore, we have

$$
\begin{aligned}
\frac{d}{dt}|\langle f\rangle_\nu-1| = &-(|1-\langle f\rangle_\nu|\pm\epsilon_1(t))\int\Big|\frac{d(x,1)-d(x,\langle f\rangle_\nu-\epsilon_1(t))}{((1+\epsilon_1(t))-\langle f\rangle_\nu)}\Big| f d\nu(x) \\
&-(|1-\langle f\rangle_\nu|\pm\epsilon_2(t))\int\Big|\frac{B(x,1)-B(x,\langle f\rangle_\nu-\epsilon_2(t))}{((1+\epsilon_2(t))-\langle f\rangle_\nu)}\Big| f(t,x)U(x)dx,
\end{aligned}
$$

and we find that (3.12) is satisfied. Hence the lemma is proved. ∎

PROPOSITION 3.6. *Assume (3.1) - (3.4) then we have that, for* $C_2 < \frac{Ck}{2C_3\tilde{K}}$,

$$
\limsup_{t\to\infty}|\langle f\rangle_\nu - S_i| \le r\max_i \limsup_{t\to\infty}|S_i-1|.
$$

Proof. Using (3.1) and Cauchy-Schwarz inequality we find that

$$
\begin{aligned}
&\frac{d}{dt}\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x) \\
&\quad \le -\tfrac{2C}{C_3}\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x) - \phi(0)\int [f(t,x)-f(t,0)]^2 B(x,1)U(x)dx \\
&\qquad +2\phi(0)(\int |f(t,0)-f(t,x)|^2 d\nu(x))^{1/2} C_1|S_2-1|\int f\tfrac{|B(x,1)-B(x,S_2(t))|}{C_1}U(x)dx \\
&\qquad +2C_2(\int f^2 d\nu(x))^{1/2}|S_1-1|(\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x))^{1/2} \\
&\quad \le -\tfrac{2C}{C_3}\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x) - C_0\int [f(t,x)-f(t,0)]^2 d\nu(x) \\
&\qquad +2C_1(\int |f(t,0)-f(t,x)|^2 d\nu(x))^{1/2}|S_2-1|\tilde{K} \\
&\qquad +2C_2(\int f^2 d\nu(x))^{1/2}|S_1-1|(\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x))^{1/2} \\
&\quad \le -\tfrac{C}{C_3}\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x) \\
&\qquad -\sqrt{\tfrac{C_0 C}{C_3}}\sqrt{\int [f(t,x)-f(t,0)]^2 d\nu(x)\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x)} \\
&\qquad +2C_1(\int |f(t,0)-f(t,x)|^2 d\nu(x))^{1/2}|S_2-1|\tilde{K} \\
&\qquad +2C_2(\int f^2 d\nu(x))^{1/2}|S_1-1|(\int (f-\langle f\rangle_\nu)^2(t,x)d\nu(x))^{1/2},
\end{aligned}
$$

and define $w(t) := \|(f-\langle f\rangle_\nu)(t,.)\|^2_{L^2_{d\nu}}$, then we have

$$
\begin{aligned}
\frac{d}{dt}w(t) \le -\sqrt{\frac{C_0 C}{C_3}}\Big(\int |f(t,0)-f(t,x)|^2 d\nu(x)\Big)^{1/2}\Big[(w(t))^{1/2} - \frac{2C_1\sqrt{C_3}\tilde{K}}{\sqrt{C_0 C}}|S_2-1|\Big] \\
-\frac{C}{C_3}(w(t))^{1/2}\Big[(w(t))^{1/2} - \frac{2C_2C_3(\int f^2 d\nu(x))^{1/2}}{C}|S_1-1|\Big].
\end{aligned}
$$

Using the similar argument in Lemma 3.5, we get

$$\limsup_{t\to\infty}\sqrt{w(t)}\le\max\Big(\frac{2C_1\sqrt{C_3}\tilde{K}}{\sqrt{C_0C}}\limsup_{t\to\infty}|S_2-1|,\limsup_{t\to\infty}\frac{2C_2C_3(\int f^2d\nu(x))^{1/2}}{C}|S_1-1|\Big).$$

Thus, we find that

$$\limsup_{t\to\infty}\|(f-\langle f\rangle_\nu)(t,.)\|^2_{L^2_{d\nu}}\le\max\Big(\limsup_{t\to\infty}\big(\frac{2C_2C_3\tilde{K}}{kC}\big)^2\|f(t,.)\|^2_{L^2_{d\nu}},\Big(\frac{2C_1\sqrt{C_3}\tilde{K}^2}{k\sqrt{C_0C}}\Big)^2\Big),$$

and for $C_2<\frac{Ck}{2C_3\tilde{K}}$, we have

$$\limsup_{t\to\infty}\|f\|_{L^2_{d\nu}}\le\max\Big(\frac{\tilde{K}}{\sqrt{(1-(\frac{2C_2C_3\tilde{K}}{kC})^2)}},\sqrt{\tilde{K}^2+\Big(\frac{2C_1\sqrt{C_3}\tilde{K}^2}{k\sqrt{C_0C}}\Big)^2}\Big),$$

therefore

$$\begin{aligned}\limsup_{t\to\infty}\sqrt{w(t)}\le\max\Big(&\frac{2C_1\sqrt{C_3}\tilde{K}}{\sqrt{C_0C}}\limsup_{t\to\infty}|S_2-1|,\\ &2C_2\frac{\tilde{K}}{C\sqrt{(1-(\frac{2C_2C_3\tilde{K}}{kC})^2)}}\limsup_{t\to\infty}|S_1-1|,\\ &\frac{2C_2}{C}\sqrt{\tilde{K}^2+\Big(\frac{2C_1\sqrt{C_3}\tilde{K}^2}{k\sqrt{C_0C}}\Big)^2}|S_1-1|\Big).\end{aligned}$$

Since we have

$$|\langle f\rangle_\nu-\langle f\rangle_{\psi_iU}|=|\langle f-\langle f\rangle_\nu\rangle_{\psi_iU}|\le\frac{1}{k}\sqrt{\langle(f-\langle f\rangle_\nu)^2\rangle_\nu},$$

we find that $\limsup_{t\to\infty}|\langle f\rangle_\nu-S_i|\le r\max_i\limsup_{t\to\infty}|S_i-1|$ is satisfied. □

Proof. (of Theorem 3.2) To prove this theorem it is enough to prove that $\|(f-\langle f\rangle_\nu)^2(t,x)\|_{L^2_{d\nu}}=0$, as $t\to\infty$. Using Lemmas 3.3–3.5 and Proposition (3.6), we have

$$\limsup_{t\to\infty}|\langle f\rangle_\nu-1|\le\limsup_{t\to\infty}|\langle f\rangle_\nu-S_i|\le r\limsup_{t\to\infty}|S_i-1|,$$

and finally

$$\limsup_{t\to\infty}|S_i-1|\le\limsup_{t\to\infty}|\langle f\rangle_\nu-1|+\limsup_{t\to\infty}|\langle f\rangle_\nu-S_i|\le 2r\limsup_{t\to\infty}|S_i-1|,$$

with $r<1/2$ (see Lemma 3.6). Therefore $S_i\to_{t\to\infty}1$, $\langle f\rangle_\nu\to_{t\to\infty}1$ and finally we have

$$\limsup_{t\to\infty}\|(f-\langle f\rangle_\nu)^2(t,x)\|_{L^2_{d\nu}}=0.$$

Hence the theorem is proved. □

4. Numerical Simulations. In this section, we present some examples to illustrate the result of Section 3. We have considered finite difference schemes with uniform step size to perform our numerical simulations. Let Δt and Δx be the step size for time and age respectively. We have used $\Delta t = 0.004$ and $\Delta x = 0.1$ for our numerical simulations.

Example 1 (convergence of the solution to the steady state)
Assume that the vital rates, initial data and the competition weights are given by

$$d(x,S) = (1+e^{-x})S,\ B(x,S) = 2 + e^{-x-S},\ u_0(x) = e^{-x},\ \psi_1 = \psi_2 = 4.1376\chi_{[0,22]}.$$

In Figure 2, we compare the normalized solution of (1.1)–(1.2) with the steady state (normalized) at $t = 20$ and the absolute error plot is also given. We observe that the normalized solution of (1.1)–(1.2) is in good agreement to the normalized vector of its steady state. In Figure 4, we plot the solution of (1.1)–(1.2) at $x = 10$ and we notice that the solution converges for a longtime. We have also given the plots of (U, ϕ) in Figure 3.

Example 2 (behavior of $S(.)$ when C or ψ_i's varies)
In this example, we assume that the birth and death rates can be written as a product of two terms: one depends of the age x and the other depends on the weighted population S. The vital rates and initial data are given by:

$$d(x,S) = \big(1 + \frac{x}{10}\big)\big(\frac{S}{10} + 0.1\big),\ B(x,S) = 10e^{-80(x-2)^2 - 1.1S},\ u_0(x) = e^{-x}.$$

In Figures 5 and 6, we have given the plot of $S(.)$ with $C = 0.15$ and $C = 0.00001$ using $\psi_1 = \psi_2 = 1\chi_{[0,22]}$, we notice that the oscillations will disappear as C increases. We also observe that the $S(.)$ converges for large value of the diffusion constant in large time, where as it does not converge for small values of the diffusion constant. Contrary, for the fixed diffusion constant $C = 0.00001$ from the Figures (6)–(9), we notice that the $S(.)$ also converges for large time when ψ_i varies (keeping $supp(\psi_i)$ fixed).

5. Conclusions. We have proved the existence and uniqueness result for a nonlocal nonlinear renewal equation with diffusion and its corresponding steady state equation also. We have used the fixed point techniques to prove the results. We have established the convergence of the solution to its steady state for large time. We took the help of GRE inequality and the Poincaré Writinger type inequality for unbounded domain. The assumptions on the vital rates *i.e.*, assumptions (3.2)–(3.4) are strong to prove the convergence of the solution, we would try to minimize the assumptions on the birth and death rates to prove the convergence part. Some numerical simulations are given to make sure that the solution converges. These simulations shows that the normalized solution of (1.1)–(1.2) converges to the normalized vector of its steady state for large time. We have also given an example where the $S(.)$ oscillates and does not converge for the small values of the diffusion constant C and the oscillations vanishes and converges when C is large or ψ_i's are moving towards $+\infty$ (keeping $supp(\psi_i)$ fixed). We would like to work on the bifurcation analysis of the diffusion constant C, local convergence and linear stability around the steady state of the solution of the equation (1.1)–(1.2).

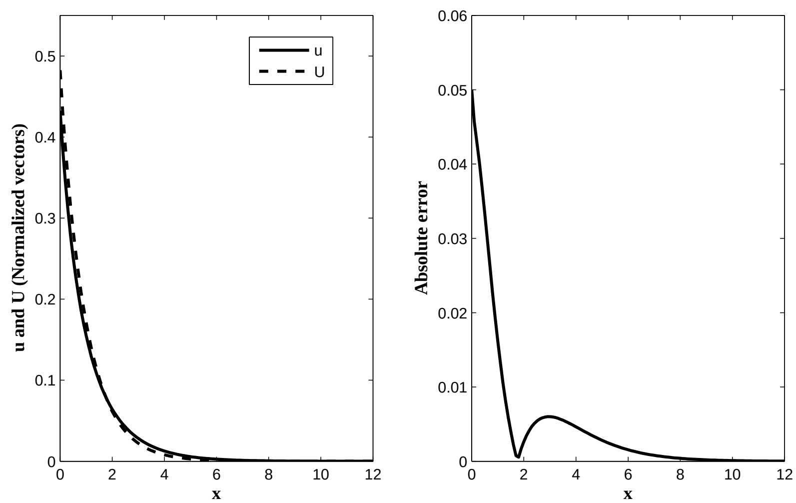

Fig. 2: left:Solution of (1.1)–(1.2) at $t = 20$. Right: Absolute error.

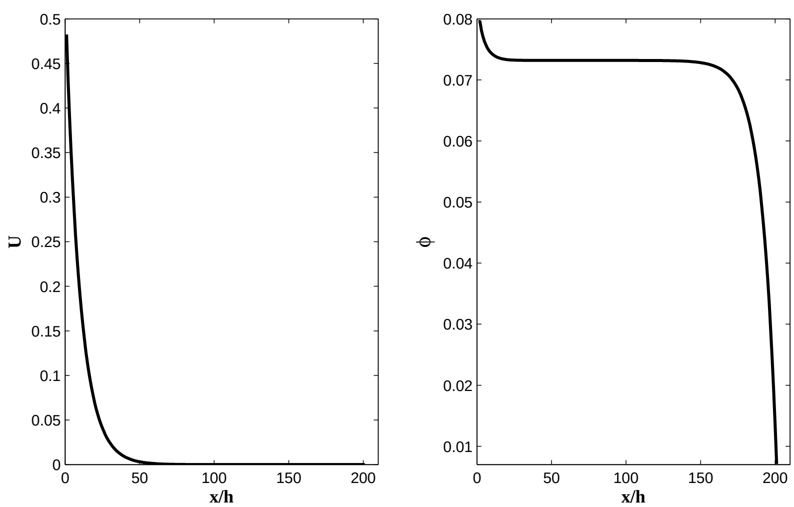

Fig. 3: (U, ϕ) $-$ Normalized vectors.

Appendix. In this section, we give alternative poof for the Proposition 2.3 under different assumptions on the vital rates and detailed calculations for the equation (3.7) in Lemma 3.3.

THEOREM 5.1. *Assume that* $0 < B_m \leq B(.,.) \leq B_M$ and $0 < d_m \leq d(.,.) \leq d_M$, *where* B_m, B_M, d_m, d_M *are positive consists, then there is unique solution to the differential equations (1.3) and (1.4).*

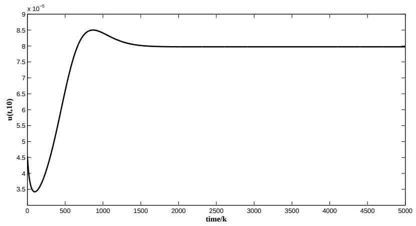

Fig. 4: Solution of (1.1)–(1.2) at $x = 10$.

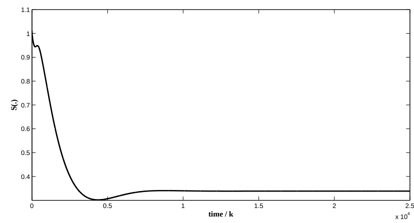

Fig. 5: $C = 0.15,\ \psi_1 = \psi_2 = 1\chi_{[0,22]}$.

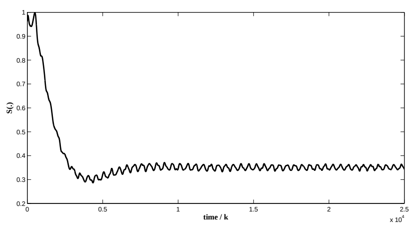

Fig. 6: $C = 0.00001,\ \psi_1 = \psi_2 = 1\chi_{[0,22]}$.

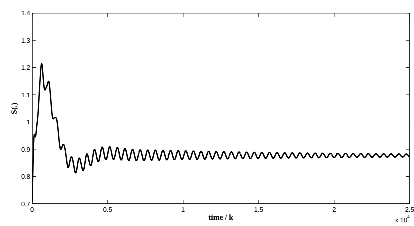

Fig. 7: $C = 0.00001,\ \psi_1 = \psi_2 = 1\chi_{[0.5,22.5]}$.

Proof. First we consider the case of bounded domain, *i.e.*,

$$\begin{cases} U_R'(x) + d(x,\bar{S}_1)U_R(x) = CU_R''(x), \qquad x \in (0,R), \\ U_R(0) - CU_R'(0) = \displaystyle\int_0^R B(x,\bar{S}_2)U_R(x)dx, \\ \displaystyle\int_0^R U_R(x)dx = 1, \text{ and } U_R(R) = 0. \end{cases} \tag{5.1}$$

and its adjoint equation is given by

$$\begin{cases} -\phi_R'(x) + d(x,\bar{S}_1)\phi_R(x) = C\phi_R''(x) + \phi_R(0)B(x,\bar{S}_2), \qquad x \in (0,R), \\ \phi_R'(0) = 0, \\ \displaystyle\int_0^R \phi_R(x)U(x)dx = 1, \text{ and } \phi_R(R) = 0. \end{cases} \tag{5.2}$$

Denote $U_R,\ \phi_R$ be the solutions of (5.1) and (5.2) respectively. Existence of $U_R,\ \phi_R$ can be obtained by using the method of subsolution and supersolution (see [19]).
Step 1: In this step, we prove the existence of solution to (1.3). Multiply U_R on both sides to equation (5.1) and integrate from 0 to R and use integration by parts to get

$$C\int_0^R |U_R'|^2 dx + \int_0^R d(x,\bar{S}_1)|U_R|^2 dx \le \frac{1}{2}B_M^2.$$

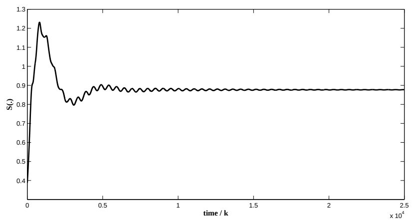

Fig. 8: $C = 0.00001$, $\psi_1 = \psi_2 = 1\chi_{[1,23]}$.

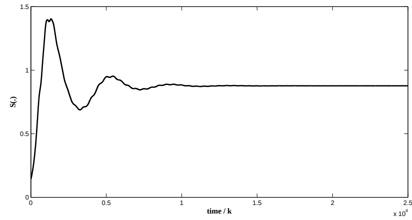

Fig. 9: $C = 0.00001$, $\psi_1 = \psi_2 = 1\chi_{[2,24]}$.

Now consider

$$C\int_0^R |U_R'|^2dx + d_m\int_0^R |U_R|^2dx \le C\int_0^R |U_R'|^2dx + \int_0^R d(x,\bar S_1)|U_R|^2dx \le \frac{1}{2}B_M^2.$$

Hence, $U_R \in W^{1,2}(\mathbb{R}^+)$. Set $U_R(x) = 0$ in $x > R$, then there exists a subsequence (still denote them by U_R) we get the existence of U which is the solution of (1.3) such that $U_R \rightharpoonup U$ in $W^{1,2}(\mathbb{R}^+)$. Uniqueness follows from the standard arguments (see Section 2).

Step 2: In this step, we prove the existence of solution to (1.4). Multiply $e^{\frac{x}{c}}$ on both sides to equation (5.2) and after rewriting we get

$$-\big(\phi_R'(x)e^{\frac{x}{c}}\big)' + \frac{1}{C}d(x,\bar S_1)\phi_R(x)e^{\frac{x}{c}} = \frac{1}{C}\phi_R(0)B(x,\bar S_2)e^{\frac{x}{c}},$$

multiplying the above equation with $g(x)$ and integrate from 0 to R to get

$$1 + \int_0^R \frac{1}{C}d(x,\bar S_1)\phi_R(x)e^x g(x)dx = \phi_R(0)\Big(1 + \frac{1}{C}\int_0^R B(x,\bar S_2)e^x g(x)dx\Big),$$

where g solves the differential equation

$$\begin{cases} -(e^{\frac{x}{c}}g'(x))' = U_R(x),, & x \in (0,R), \\ g(R) = 0,\ g'(0) = 1. \end{cases} \tag{5.3}$$

It is easy to observe that $e^{\frac{x}{c}}g(x) \le U_R(x)$. Using (1.7) we get $\frac{C+d_m}{C+B_M} \le \phi_R(0) \le 1 + \frac{d_M}{C}$.

Multiplying $\phi_R(x)U_R(x)$ to equation (5.2) and integrate from 0 to R and use integration by parts to get

$$C\int_0^R (\phi_R'(x))^2U_R(x)dx + \frac{1}{2}\int_0^R d(x,\bar S_1)(\phi_R(x))^2U_R(x)dx$$
$$= \phi_R(0)\int_0^R B(x,\bar S_2)\phi_R(x)U_R(x)dx - \frac{1}{2}\phi_R^2(0)\int_0^R B(x,\bar S_2)U_R(x)dx.$$

Therefore we have

$$C\int_0^R (\phi_R'(x))^2U_R(x)dx + \frac{d_m}{2}\int_0^R (\phi_R(x))^2U_R(x)dx \le \phi_R(0)B_M \le (1+\frac{d_M}{C})B_M.$$

It is easy to show that $\|\phi_R\| \in L_\infty(\mathbb{R}^+)$ (see [1]). By similar analysis as in Step 1, we get the existence of ϕ which is the solution of (1.4). From the Lebesgue theorem we get $\int_0^\infty U(x)\phi(x)dx = 1$. ∎

REMARK 5.1. *In equation (5.1), even if* $\int_0^R U_R(x)dx = 1$ *replaced by* $\int_0^R U_R(x)dx < \infty$ *the proof still works by modifying the constants accordingly.*

LEMMA 5.2. *Let* u, U, ϕ *be solution to equation (1.1), (1.3) and (1.4) respectively. Let the entropy is defined by*

$$\mathcal{H}(t) := \int_0^\infty H(u/U)U\phi dx,$$

where H *is a convex function. Then we have*

$$\frac{d}{dt}\mathcal{H}(t) = CD^H_{diff}(u) + D^H_{ren}(u) + Remainder,$$

where the entropy dissipation due to diffusion and the renewal terms are

$$D^H_{diff}(u) = -\int_0^\infty H''\Big(\frac{u(t,x)}{U(x)}\Big)U(x)\phi(x)\Big(\frac{\partial}{\partial x}\frac{u(t,x)}{U(x)}\Big)^2 dx,$$

$$\begin{aligned}\frac{D^H_{ren}(u)}{\phi(0)} = -\int_0^\infty \Big\{ &H\Big(\frac{u(t,x)}{U(x)}\Big) - H\Big(\frac{u(t,0)}{U(0)}\Big) \\ &- H'\Big(\frac{u(t,0)}{U(0)}\Big)\Big[\frac{u(t,x)}{U(x)} - \frac{u(t,0)}{U(0)}\Big]\Big\} \int_0^\infty B(x,\bar{S}_2)U(x)dx.\end{aligned}$$

$$\begin{aligned}Remainder = \phi(0)\int_0^\infty &H'\Big(\frac{u(t,0)}{U(0)}\Big)\big[B(x,S_2) - B(x,\bar{S}_2)\big]udx \\ &- \int_0^\infty H'(\frac{u}{U})\big[d(x,S_1) - d(x,\bar{S}_1)\big]u\phi dx.\end{aligned}$$

Proof. Consider

$$\begin{aligned}\Big(H(\frac{u}{U})\phi U\Big)_t + \Big(H(\frac{u}{U})\phi U\Big)_x = &CH'(\frac{u}{U})[\frac{Uu_{xx} - uU_{xx}}{U}]\phi \\ &+ H(\frac{u}{U})(\phi U)_x - H'(\frac{u}{U})\big[d(x,S_1) - d(x,\bar{S}_1)\big]u\phi.\end{aligned}$$

then we get

$$\frac{d}{dt}\ \mathcal{H}(t) = L + I_1 + I_2 + I_3 + I_4.$$

where

$$L = H(\frac{u(t,0)}{U(0)})\phi(0)U(0), \qquad I_1 = C\int_0^\infty H'(\frac{u}{U})\phi u_{xx},$$

$$I_2 = -C\int_0^\infty H'(\frac{u}{U})\frac{u}{U}\phi U_{xx}, \qquad I_3 = \int_0^\infty H(\frac{u}{U})(\phi U)_x,$$

$$I_4 = -\int_0^\infty H'(\frac{u}{U})\big[d(x,S_1) - d(x,\bar{S}_1)\big]u\phi.$$

First we Consider

$$\begin{aligned}
I_1 &= C\int_0^\infty H'(\frac{u}{U})\phi u_{xx}dx \\
&= -C\int_0^\infty \big\{H'(\frac{u}{U})\phi\big\}_x u_x dx - CH'(\frac{u(t,0)}{U(0)})\phi(0)u_x(t,0) \\
&= -C\int_0^\infty H''(\frac{u}{U})(\frac{u}{U})_x\phi u_x dx - C\int_0^\infty H'(\frac{u}{U})\phi_x u_x dx - CH'(\frac{u(t,0)}{U(0)})\phi(0)u_x(t,0) \\
&= -C\int_0^\infty \Big[H''(\frac{u}{U})(\frac{u}{U})_x\phi u_x + H'(\frac{u}{U})\phi_x u_x\Big]dx + C_1.
\end{aligned}$$

where $\quad C_1 = -CH'(\frac{u(t,0)}{U(0)})\phi(0)u_x(t,0)$.
Now consider

$$\begin{aligned}
I_2 &= -C\int_0^\infty H'(\frac{u}{U})\frac{u}{U}\phi U_{xx}dx \\
&= C\int_0^\infty \big\{H'(\frac{u}{U})\frac{u}{U}\phi\big\}_x U_x dx + C\phi(0)H'(\frac{u(t,0)}{U(0)})(\frac{u(t,0)}{U(0)})U_x(0) \\
&= C\int_0^\infty \Big[H''(\frac{u}{U})(\frac{u}{U})_x(\frac{u}{U})\phi U_x + H'(\frac{u}{U})U_x(\frac{u}{U}\phi)_x\Big]dx + C_2.
\end{aligned}$$

where $\;C_2 = C\phi(0)H'(\frac{u(t,0)}{U(0)})(\frac{u(t,0)}{U(0)})U_x(0)$.
Adding I_1 and I_2 to get

$$\begin{aligned}
I_1 + I_2 &= C\int_0^\infty H''(\frac{u}{U})(\frac{u}{U})_x\phi U(-\frac{u_x}{U} + \frac{uU_x}{U^2})dx + C\int_0^\infty H'(\frac{u}{U})\Big[(\frac{u}{U})_x\phi U_x \\
&\qquad + (\tfrac{u}{U})U_x\phi_x - u_x\phi_x\Big]dx + C_1 + C_2 \\
&= CD^H_{diff} + C\int_0^\infty H'(\frac{u}{U})\Big[(\frac{u}{U})_x\phi U_x + \phi_x U\big(-\frac{u_x}{U} + \frac{uU_x}{U^2}\big)\Big]dx + C_1 + C_2 \\
&= CD^H_{diff} + C\int_0^\infty H'(\frac{u}{U})(\frac{u}{U})_x\ [\phi U_x - \phi_x U]dx + C_1 + C_2.
\end{aligned}$$

Now consider

$$\begin{aligned}
I_3 &= \int_0^\infty H(\frac{u}{U})\ \{C\phi U_{xx} - CU\phi_{xx} - B(x,\bar{S}_2)U\phi(0)\}dx \\
&= C\int_0^\infty H'(\frac{u}{U})(\frac{u}{U})_x\ [-\phi U_x + \phi_x U]dx \\
&\qquad - \int_0^\infty H(\frac{u}{U})B(x,\bar{S}_2)U\phi(0)dx - C\phi(0)H(\frac{u(t,0)}{U(0)})U_x(0) \\
&= C\int_0^\infty H'(\frac{u}{U})(\frac{u}{U})_x\ [-\phi U_x + \phi_x U]dx - \int_0^\infty H(\frac{u}{U})B(x,\bar{S}_2)U\phi(0)dx + C_3.
\end{aligned}$$

where $\quad C_3 = -C\phi(0)H\big(\frac{u(t,0)}{U(0)}\big)U_x(0).$

Now we consider

$$\begin{aligned} C_1 + C_2 \quad &= H'\big(\tfrac{u(t,0)}{U(0)}\big)\phi(0)\big[-Cu_x(t,0) + \big(\tfrac{u(t,0)}{U(0)}\big)CU_x(0)\big] \\ &= H'\big(\tfrac{u(t,0)}{U(0)}\big)\phi(0)\Big[-u(t,0) + \textstyle\int_0^\infty B(x,S_2)u(x)dx \\ &\qquad\qquad + u(t,0)\big(1 - \tfrac{1}{U(0)}\textstyle\int_0^\infty B(x,\bar{S}_2)u(x)dx\big)\Big] \\ &= \phi(0)H'\Big(\tfrac{u(t,0)}{U(0)}\Big)\Big\{\textstyle\int_0^\infty B(x,S_2)u(x)dx - \tfrac{u(t,0)}{U(0)}\textstyle\int_0^\infty B(x,\bar{S}_2)U(x)dx\Big\}. \end{aligned}$$

Now we consider

$$\begin{aligned} L + C_3 \quad &= H\big(\tfrac{u(t,0)}{U(0)}\big)\phi(0)U(0) - C\phi(0)H\big(\tfrac{u(t,0)}{U(0)}\big)U_x(0) \\ &= H\big(\tfrac{u(t,0)}{U(0)}\big)\phi(0)\textstyle\int_0^\infty B(x,\bar{S}_2)u(x)dx. \end{aligned}$$

Hence after rewriting we get

$$\frac{d}{dt}\int_0^\infty H(u/U)U\phi dx = CD^H_{diff}(u) + D^H_{ren}(u) + Remainder.$$

Now take $H(x) = x^2$ and use (3.6) to get (3.7). $\square$

Acknowledgements. Bhargav Kumar Kakumani has been supported by Institut Camille Jordan and CSIR.

REFERENCES

[1] B. Abdellaoui, T.M. Touaoula, Decay solution for the renewal equation with diffusion, Nonlinear Differ. Equ. Appl. (Nodea), 17 (2010), pp. 271-288.

[2] O. Arino, E. Sanchez, A survey of cell population dynamics, J. Theor. Med., 1 (1997), pp. 35-51.

[3] B. Basse, B.C. Baguley, E.S. Marshall, W.R. Joseph, B. Van Brunt, G. Wake, D.J.N. Wall, A mathematical model for analysis of the cell cycle in cell lines derived from human tumors, Journal of Mathematical Biology, 47(4) (2003), pp. 295-312.

[4] A. Benabdallah, M. Henry, Approximation by Diffusion of Renewal Equations, SeMA Journal, 56 (2011), pp. 5-34.

[5] J.W. Brewer, The age-dependent eigenfunctions of certain Kolmogorov equations of engineering, economics, and biology, Applied Mathematical Modeling, 13 (1989), pp. 47-57.

[6] J.M. Cushing, An Introduction to Structured Population Dynamics, SIAM, Philadelphia, 1998.

[7] A. Goldbeter, Biochemical oscillators and cellular rhythms, Cambridge University Press, Cambridge, 1997.

[8] M. Iannelli, Gabriela Marinoschi, Approximation of a population dynamics model by parabolic regularization, Mathematical methods in the Applied Sciences, 36 (2013), pp. 1229-1239.

[9] M. Iannelli, Mathematical Theory of Age-Structured Population Dynamics, in: Applied Mathematics Monograph C.N.R., vol. 7, Giardini editori estampatori, Pisa, 1995.

[10] M. Iannelli, J. Ripoll, Two-sex age structured dynamics in a fixed sex-ratio population, Nonlinear Analysis: Real World Applications, 13 (2012), pp. 2562-2577.

[11] B.K. Kakumani and S.K. Tumuluri, On a nonlinear renewal equation with diffusion, Mathematical methods in the Applied Sciences, 2015 (published online).

[12] B.K. Kakumani and S.K. Tumuluri, Extinction and blow-up phenomena in a nonlinear gender structured population model, Nonlinear analysis: Real World Applications, 28 (2016), pp. 290–299.

[13] P. Michel, General Relative Entropy in a nonlinear McKendrick model, AMS : Contemporary Mathematics 429, Stochastic Analysis and Partial Differential Equations (Gui-Qiang Chen, Elton Hsu, and Mark Pinsky), (2007), pp. 205–232.

[14] P. Michel, Optimal proliferation rate in a cell division model, Mathematical Modelling of Natural Phenomena, 1(2) (2006), pp. 23-44.
[15] P. Michel, Fitness optimization in a cell division model, Comptes Rendus Mathmatique, 341(12) (2005), pp. 731-736.
[16] P. Michel, S. Mischler, B. Perthame, General entropy equations for structured population models and scattering, C.R. Acad. Sci. Paris, 338(1) (2004), pp. 697-702.
[17] P. Michel, S. Mischler, B. Perthame, General relative entropy inequality: an illustrations on growth models, J. Math. Pures Appl., 84(9) (2005), pp. 1235-1260.
[18] P. Michel, T.M. Touaoula, Asymptotic behavior for a class of the renewal nonlinear equation with diffusion, Mathematical methods in the Applied Sciences, 36(3) (2012), pp. 323–335.
[19] C.V. Pao, Nonlinear Parabolic and Elliptic equations, Plenum, New York, 1992.
[20] B. Perthame, Transport Equations in Biology, LN Series Frontiers in Mathematics, Birkhäuser, 2007.
[21] B. Perthame, S.K. Tumuluri, Nonlinear renewal equations, in: N. Bellomo, M. Chaplain, E.De Angelis (Eds.), Selected Topics on Cancer Modeling Genesis - Evolution - Immune Competition - Therapy, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, 2007, pp. 65-96.
[22] H.R. Thieme, Mathematics in Population Biology, Princeton University Press, Princeton, NJ, 2003.
[23] S.K. Tumuluri, Steady state analysis of a non-linear renewal equation, Mathematical and Computer Modeling, 53 (2011), pp. 1420-1435.
[24] G.F. Webb, Theory of non-linear age-dependent population dynamics, in: Monographs and Textbook in Pure and applied Mathematics, Vol.89, Marcel Dekker Inc., New-York Basel, 1985.